Introduction to Countability and Infinite Sets

When dealing with infinite sets such as real numbers or integers, mathematicians use the concept of **countability** to describe the size of these sets (*cardinality*).

Countable vs Uncountable Sets

- **Countable sets** are those whose elements can be matched one-to-one with the set of natural numbers (e.g., integers, rational numbers).
- **Uncountable sets** are too large to be put into such a one-to-one correspondence (e.g., real numbers between 0 and 1).

Decimal Expansions and Counting Finite n-Digit Numbers

If we consider decimal numbers between 0 and 1 with exactly *n* digits after the decimal point:

- For $n = 1$ digit, possible numbers are 0.0, 0.1, ..., 0.9, so total = 10 numbers.
- For $n = 2$ digits, from 0.00 to 0.99, total = $10^2 = 100$ numbers.
- For general *n*, total numbers = 10^n.

This set is finite for any finite *n*.

As n approaches infinity:

Considering all infinite decimal expansions (numbers with infinitely many decimal digits after the decimal point) defines the set of all real numbers between 0 and 1. This set is uncountable as proved by Cantor's diagonal argument.

Cantor's Diagonal Argument: Why the Real Numbers are Uncountable

Cantor's ingenious argument shows that any purported list of real numbers between 0 and 1 can be proven incomplete by constructing a new real number that differs from every number in the list at least at one decimal place.

This definitively shows the set of infinite decimal expansions (hence reals) is uncountable, despite the fact that for each finite digit count *n*, the decimals are countable.

Prime Numbers and Countability

The set of prime numbers is **infinite but countable** because we can list prime numbers as a sequence analogous to natural numbers. As your sequence `c(n+2) = -1 + 2^{c(n+1)}` grows extremely quickly and involves large exponents, it does not contradict the prime set countability:

- Even though primes may grow fast, the collection of all primes can be put in one-to-one correspondence with natural numbers, making the set countable.
- Recall: *countable infinite* includes many different types of infinite sets, including sets that grow very rapidly.

Higher-Dimensional Spaces and Counting Points

When considering multi-dimensional intervals such as in 2D or n-dimensional hypercubes with points between 0 and 1 in each dimension, the number of points behaves differently:

- For each dimension *d*, the number of points with finite decimal expansions having *n* digits after the decimal is roughly 10^{n*d} (not $10^{n\hat{}2}$).
- As *n* goes to infinity, the set of points corresponds to the real numbers in d-dimensional space, which is also uncountable.

Clarifying Your Key Questions

1. **Is 10^n countable or uncountable as n → ∞?**
   The set of all finite decimal numbers with n digits is finite for any fixed n, countable when unioned over all finite n, but the set of infinite decimal expansions is uncountable.
2. **Is Cantor's diagonal argument false logic or taboo?**
   No. Cantor's diagonal argument is mathematically rigorous, widely accepted, and underpins modern set theory. It is not false or taboo but foundational to understanding infinity.
3. **Are rational numbers countable?**
   Yes, rational numbers are countable via pairing numerator and denominator with natural numbers and arranging them in a sequence.
4. **Disagreement with Cantor's views:**
   The distinctions come from the difference between finite decimal expansions, infinite expansions, and how infinite sequences are counted. It's crucial to distinguish between these carefully to avoid misunderstandings.

Summary

- Finite decimal expansions (fixed n digits) form finite/countable sets.
- Infinite decimal expansions form uncountable sets due to Cantor's diagonal argument.
- Prime numbers, even though infinite and growing fast, are countable.
- Higher dimensional intervals increase the count exponentially for finite digits but remain uncountable for infinite expansions.
- Cantor's arguments remain valid and are core to current mathematics.

If you would like, I can provide more detailed examples or proofs regarding any of these points.