Lesson Plan: Design a Medieval Fortress

Subject: Mathematics (Applied Algebra, Geometry, and Trigonometry)

Grade Level: 10 (with extensions)

Focus: This lesson blends concepts from the Australian Curriculum (ACARA), Art of Problem Solving (AOPS), and Beast Academy's creative problem-solving approach with a student's interest in medieval infrastructure.

Materials Needed:

- Graph paper (or a digital drawing tool/app)
- Pencil and eraser
- Ruler
- Scientific calculator
- Access to the internet for optional research on castle design

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1. Learning Objectives

By the end of this lesson, you will be able to:

- Apply linear equations to model real-world constraints like budget and materials.
- Use formulas for perimeter, area, surface area, and volume to design geometric structures.
- Model a real-world scenario (a catapult launch) using a quadratic function.
- Apply basic right-triangle trigonometry (SOH CAH TOA) to solve problems involving angles and distances.
- Create a mathematical justification for a creative design project.

Curriculum Alignment (ACARA - Year 10): ACMNA239 (Linear equations), ACMNA241 (Quadratic equations), ACMMG242 (Surface area and volume), ACMMG245 (Trigonometry).

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2. The Scenario: The Royal Engineer's Quest

Welcome, Royal Engineer! The King of Arithmetica has commissioned you to design a new, small fortress to protect the kingdom's northern border. However, the treasury is not infinite! You have a strict budget and a limited supply of stone. Your quest is to design a fortress that is both defensible and affordable by using your mastery of mathematics.

Your Constraints:

- **The Royal Treasury:** You have a total budget of **20,000 gold pieces**.
- **Building Costs:**
  - Outer Wall Stone: **10 gold pieces** per cubic meter (m^3).
  - Tower Stone: **12 gold pieces** per cubic meter (m^3) (it's higher quality).
- **The Land:** You have a flat, square plot of land that is 100 meters by 100 meters.

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3. Main Activity: The Design Blueprint (Step-by-Step)

Part 1: The Curtain Wall (Linear Equations & Area)

Your first task is to design the outer defensive wall. For simplicity, we will assume it is a rectangle.

1. **Choose Dimensions:** Decide on the length (L) and width (W) of your rectangular fortress. Remember it must fit on your 100m x 100m plot.
2. **Calculate Perimeter:** The perimeter of your fortress is **P = 2L + 2W**. This is the total length of your wall.
3. **Determine Wall Volume:** Let's say your wall must be **8 meters high** and **3 meters thick**. The total volume of stone needed for the wall is **V_{wall} = P × 8m × 3m**.
4. **Calculate Cost:** The cost of your wall is **C_{wall} = V_{wall} × 10 gold**.
   Challenge Question: Can you write a single linear equation that connects the cost (C), length (L), and width (W)? This will help you quickly see if your design is over budget.
5. **Action:** On your graph paper, draw the outline of your wall. Label the dimensions and write down the total volume and cost. Make sure your wall cost does not exceed your 20,000 gold budget!

Part 2: The Watchtowers (Volume & Trigonometry)

A wall is useless without towers for archers. You must add **four identical cylindrical towers**, one at each corner of your fortress wall.

1. **Design Your Towers:** Your towers must be **15 meters high**. You need to choose the radius (r) of the towers. A larger radius makes them stronger but also more expensive.
2. **Calculate Tower Volume:** The volume of one cylindrical tower is **$V_{tower} = \pi r^2 h$**, where h = 15m. Calculate the volume for all four towers (4 × V_{tower}).
3. **Calculate Tower Cost:** The cost of your towers is **C_{towers} = (4 × V_{tower}) × 12 gold**.
4. **Budget Check:** Calculate your total project cost so far: **Total Cost = C_{wall} + C_{towers}**. Is it under 20,000 gold? If not, you must adjust your wall dimensions or tower radius. This is the kind of optimization real engineers do!
5. **Archer's Advantage (Trigonometry):** An archer stands on top of a 15m tower. They want to shoot an enemy at the base of the tower. What is the angle of depression they must aim at? Now, imagine they see an enemy soldier 50 meters away from the base of the tower. What is the angle of depression they must use to aim at that soldier? (Hint: Draw a right-angled triangle. You know the 'opposite' side (height) and the 'adjacent' side (ground distance). Use $\tan^{-1}$).
6. **Action:** Add the towers to your graph paper blueprint. Label the radius. Calculate and record your total cost and the angles of depression you found.

Part 3: Defending Against a Catapult (Quadratic Equations)

An enemy catapult has been rolled into position 80 meters from your outer wall. Its launches follow the parabolic path given by the equation:

$$y = -0.01x^2 + 0.9x + 1$$

Where 'y' is the height of the projectile in meters and 'x' is the horizontal distance from the catapult in meters.

1. **Is Your Wall High Enough?** Your wall is located 80 meters from the catapult (so, at x = 80). Plug x=80 into the equation to find the height of the projectile when it reaches your wall. Is the projectile's height greater than your wall's height (8m)? If so, your wall is breached!
2. **Find the Maximum Height:** A parabola of the form $y = ax^2 + bx + c$ reaches its maximum height at the vertex. The x-coordinate of the vertex is found with the formula **$x = -b / 2a$**.

Calculate the x-value where the projectile is at its highest, and then plug that x-value back into the main equation to find the maximum height (y). How high does the projectile go?

3. **Find the Total Range:** Where does the projectile land? This happens when its height (y) is 0. Solve the quadratic equation **$0 = -0.01x^2 + 0.9x + 1$** for x using the quadratic formula. The positive solution will tell you how far the catapult can shoot. Will it land inside your fortress?
4. **Action:** On a separate piece of paper, sketch the path of the catapult projectile. Mark the location of your fortress wall, the maximum height of the shot, and where it lands. Write a brief conclusion: is your current fortress design safe from this catapult?

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4. Creative Extension: Your Royal Signature

Now that the basic fortress is designed, add one unique, mathematically-justified feature. Choose one of the following ideas or create your own:

- **The Moat:** Design a moat around your fortress. If it costs 5 gold pieces per cubic meter of dirt removed, calculate the cost of a moat that is 4 meters wide and 2 meters deep.
- **The Central Keep:** Design a square-based pyramid keep in the center of your fortress. Choose its base dimensions and height, and calculate its volume ($V = (1/3) \times \text{base\_area} \times \text{height}$) and cost, assuming it uses the more expensive tower stone. Make sure you don't go over budget!
- **The Angled Walls:** Research why some later medieval fortresses (star forts) used angled walls. Explain using geometric terms (e.g., eliminating blind spots, creating kill zones) how this is superior to your rectangular design.

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5. Assessment: The Royal Report

To complete your quest, you must present your design to the King. Prepare a "Royal Report" that includes:

1. **Your Final Blueprint:** Your neat, labeled drawing on graph paper.
2. **The Financials:** A clear breakdown of your calculations for the wall cost, tower cost, and total project cost.
3. **The Defense Analysis:** Your calculations and conclusions for the archer's trigonometry and the catapult's trajectory. Explain in a few sentences whether your fortress is effective.
4. **Your Signature Feature:** A drawing and the calculations for your creative extension, with a brief explanation of its purpose.

Going Further (Optional Advanced Challenges):

- **Optimization:** For a fixed perimeter, what rectangular shape encloses the most area? What if you could build a circular wall instead? Compare the area enclosed by a circular wall vs. a square wall with the same perimeter (circumference).
- **Systems of Equations:** Imagine you have a fixed budget of 20,000 gold and need to use exactly 1500 m^3 of total stone. Can you set up a system of two equations with two variables (Volume_wall and Volume_towers) to find the exact volumes of each you can afford?